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Approximation of Analytic Functions by a Class of Linear Positive Operators

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1. INTRODUCTION

Let $G \in M(\mathbb{R}^+)$, the class of complex-valued functions measurable on $(0, \infty)$, be a nonnegative function satisfying the following properties:

(i) $G(u)$ is continuous at $u=1$,

(ii) for each $\delta > 0$, $\|\chi_{\delta,1} G\|_\infty < G(1)$, and

(iii) there exist θ_1, $\theta_2 > 0$ such that $(u^{-\theta_1} + u^{\theta_2})\, G(u)$ is bounded and is in $M(\mathbb{R}^+)$.

Here $\chi_{\delta,x}$ denotes the characteristic function of the set $(0, \infty) - (x-\delta, x+\delta)$.

Such a function G is called (for our purpose) an "admissible" kernel function. The set of all admissible kernel functions will be denoted by $T(\mathbb{R}^+)$.

Let $G \in T(\mathbb{R}^+)$, $\alpha \in \mathbb{R}$, λ, $x \in \mathbb{R}^+$ and $f \in M(\mathbb{R}^+)$. We define

$$T_\lambda(f; x) = \frac{x^{\alpha-1}}{a(\lambda)} \int_0^\infty u^{-\alpha} f(u)\, G^\lambda(xu^{-1})\, du, \tag{1.1}$$

where,

$$a(\lambda) = \int_0^\infty u^{\alpha-2} G^\lambda(u)\, du,$$

whenever, the above integrals exist, (1.1) defines a class of linear positive operators.

Let Ω (>1) be a continuous function defined on $\mathbb{R}^+$. We call Ω a bounding function for a $G \in T(\mathbb{R}^+)$, if for each compact $K \subset \mathbb{R}^+$, there exist positive numbers λ_K and M_K such that

$$T_{\lambda_K}(\Omega; x) < M_K, \qquad x \in K. \tag{1.2}$$

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Clearly, for $G \in T(\mathbb{R}^+)$, $\Omega(u) = u^{-p} + u^q$ is a bounding function. The notion of a bounding function [11] enables us to obtain results in a uniform set-up, which, at the same time, are applicable for a general $G \in T(\mathbb{R}^+)$.

For a bounding function Ω, we define

$$D_\Omega = \{f\colon f \text{ is locally integrable on } (0, \infty) \\ \text{and } |f(u)| \leqslant M\Omega(u),\ u \in (0, \infty)\}.$$

Several well-known operators, such as Gamma operators of Müller [10], modified Post–Widder operators [8], Post–Widder operators [14], the operators studied in [9], etc., are particular cases of the class T_λ. This can be verified by choosing G, λ and α suitably [6].

We use T_λ, whose construction, as we will see in Section 2, depends only on the functional values on $\mathbb{R}^+$, for approximating a class of analytic functions in complex-plane. The results can also be regarded as providing us new analytic continuation methods. In this direction, we mention the works of Kantarovich [4] and Bernstein [7], who obtained results on convergence of Bernstein polynomials in the complex domain, by making an ingenious use of the Legendre's polynomials. Similar results for the Szàsz operators were obtained by Gergen *et al.* [3] by using Lagurre's polynomials. Other results in this direction have been obtained in [1, 5, 15].

We make use of the Cauchy's theorem to obtain the results on the approximation-theoretic properties of T_λ in the complex-domain.

2. Some More Definitions and Auxiliary Results

We give some more definitions and auxiliary results.

Let $G \in T(\mathbb{R}^+)$ be an analytic function, regular in the angle

$$\Lambda_\Psi = \{z = re^{i\theta}\colon r > 0, 0 \leqslant \theta < \Psi\},$$

and the property that for each $0 < \theta_0 < \Psi$ there exist θ_1, $\theta_2 > 0$ such that $g(r, \theta) = (r^{-\theta_1} + r^{\theta_2})\, G(re^{i\theta})$ is bounded on the set

$$A(\theta_0) = \{(r, \theta)\colon 0 < \theta < \theta_0, r > 0\}.$$

Let Ω be a bounding function for $G \in T(\mathbb{R}^+)$ satisfying the following requirements.

For every compact subset $K \subset \Lambda_\Psi$ there exist positive constants λ_K, M_K such that

$$M_K \geqslant |T_{\lambda_K}|(\Omega\colon z) = \frac{|z|^{\alpha-1}}{a(\lambda_K)} \int_0^\infty u^{-\alpha}\Omega(u)\,|G(zu^{-1})|^{\lambda_K}\, du.$$

In the above situation, we say that $G \in T_\Psi(\mathbb{R}^+)$ and that Ω is a Ψ-bounding function for G. It follows from the Riemann–Schwarz reflection principle [13, p. 155], that if $G \in T_\Psi(\mathbb{R}^+)$ and Ω is a Ψ-bounding function for G, then the above mentioned properties continue to hold in the reflection of Λ_Ψ, $A(\theta_0)$ and K, respectively, through the real axis.

If f is an analytic function such that for $0 < \theta < \Psi$ the limits

$$\overline{\lim_{u \to 0^+}} \sup_{0 < \theta < \theta_0} |f(ue^{i\theta})|/\Omega(u)$$

and

$$\overline{\lim_{u \to \infty}} \sup_{0 < \theta < \theta_0} |f(ue^{i\theta})|/\Omega(u)$$

exist, we say that $f \in D_\Omega^\Psi$.

If $f \in D_\Omega^\Psi$, $G \in T_\Psi(\mathbb{R}^+)$ and Ω is Ψ-bounding function then, it is obvious that given compact subset K of Λ_Ψ, T_λ $(f\colon z)$ exists and defines an analytic function regular in K, for λ sufficiently large.

Now, we give some auxiliary results useful for further developments.

LEMMA 2.1. *If $G \in T(\mathbb{R}^+)$, then*

$$\lim_{\lambda \to \infty} \frac{a(\lambda)}{G^\lambda(1)} = 0. \tag{2.1}$$

Proof. Let $\varepsilon > 0$ be arbitrarily fixed. Then we can choose a $\delta > 0$ so small that in view of the property (ii) of G

$$G^{-\lambda}(1) \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^\lambda(u)\, du < \int_{1-\delta}^{1+\delta} u^{\alpha-2}\, du < \frac{\varepsilon}{2}. \tag{2.2}$$

Let $m = \|\chi_{\delta,1} G\|_\infty$. Since there exists a λ_0 such that $a(\lambda)$ exists for all $\lambda > \lambda_0$, we have

$$G^{-\lambda}(1) \int_0^\infty u^{\alpha-2} G^\lambda(u)\, \chi_{\delta,1}(u)\, du < \frac{m^{\lambda-\lambda_0}}{G^\lambda(1)}\, a(\lambda_0).$$

Since, by the property (ii) of G, $m < G(1)$ there exists a λ_1 such that for $\lambda > \lambda_1$,

$$G^{-\lambda}(1) \int_0^\infty u^{\alpha-2} G^\lambda(u)\, \chi_{\delta,1}(u)\, du < \frac{\varepsilon}{2}. \tag{2.3}$$

Combining (2.2), (2.3) we have $0 < a(\lambda)\, G^{-\lambda}(1) < \varepsilon$ for $\lambda > \lambda_1$. But ε is arbitrary, so (2.1) follows.

LEMMA 2.2. *Let $G \in T(\mathbb{R}^+)$ and Ω be a bounding function for G. If $0<\delta<a<b<\infty$ and $f \in D_\Omega$, then*

$$\lim_{\lambda \to \infty} \lambda^k T_\lambda(f\chi_{\delta,x}; x) = 0, \tag{2.4}$$

uniformly in $x \in [a, b]$, for any $k \in \mathbb{R}^+$.

Proof. We have

$$\begin{aligned} T_\lambda(f\chi_{\delta,x}; x) &= \frac{1}{a(\lambda)} \int_0^\infty u^{\alpha-2} f(xu^{-1})\, \chi_{\delta,x}(xu^{-1})\, G^\lambda(u)\, du \\ &= \frac{1}{a(\lambda)} \left[\int_0^{x/x+\delta} + \int_{x/x-\delta}^\infty \right] u^{\alpha-2} f(xu^{-1})\, G^\lambda(u)\, du. \end{aligned}$$

Let $0<\delta<a\leqslant x\leqslant b<\infty$. We choose $\eta \in (0, 1)$ such that $b<\delta((1/\eta)-1)$. We have then,

$$\frac{x}{x+\delta} \leqslant \frac{b}{b+\delta} < 1-\eta \qquad \text{and} \qquad \frac{x}{x-\delta} \geqslant \frac{b}{b-\delta} \geqslant 1+\eta.$$

Using these inequalities and $|f(u)| \leqslant M\Omega(u)$, $u \in (0, \infty)$, we find that

$$|T_\lambda(f_{\chi_{\delta,x}} : x)| \leqslant \frac{1}{a(\lambda)} \left[\int_0^{1-\eta} + \int_{1+\eta}^\infty \right] u^{\alpha-2} \Omega(xu^{-1})\, G^\lambda(u)\, du.$$

Let λ_1 be such that $T_{\lambda_1}(\Omega; x) \leqslant M_1$ for $x \in [a, b]$ we have then for $\lambda > \lambda_1$,

$$|T_\lambda(f\chi_{\delta,x} : x)| \leqslant \frac{m_\eta^{\lambda-\lambda_1}}{a(\lambda)} \left[\int_0^{1-\eta} + \int_{1+\eta}^\infty \right] u^{\alpha-2} \Omega(xu^{-1})\, G^{\lambda_1}(u)\, du,$$

where $m_\eta = \max\{G(u) : u \leqslant 1-\eta \text{ or } u \geqslant 1+\eta\}$. We have then

$$\begin{aligned} |T_\lambda(f\chi_{\delta,x}; x)| &\leqslant \frac{m_\eta^{\lambda-\lambda_1}}{a(\lambda)} \int_0^\infty u^{\alpha-2} \Omega(xu^{-1})\, G^{\lambda_1}(u)\, du \\ &\leqslant \frac{a(\lambda_1)}{a(\lambda)} m_\eta^{\lambda-\lambda_1} T_{\lambda_1}(\Omega; x) \\ &\leqslant M_1 a(\lambda_1) \frac{m_\eta^{\lambda-\lambda_1}}{a(\lambda)}, \end{aligned}$$

for all $\lambda > \lambda_1$ and $x \in [a, b]$. Since G is continuous at 1, we can find $c \in (0, \eta)_{(0<\eta<1)}$ and $\varepsilon > 0$,

$$G(u) \geqslant (m_\eta + \varepsilon) \text{ for } u \in (1-c, 1+c).$$

We have then

$$a(\lambda)=\int_0^\infty u^{\alpha-2}G^\lambda(u)\,du$$
$$\geqslant\int_{1-c}^{1+c} u^{\alpha-2}G^\lambda(u)\,du$$
$$\geqslant(m_\eta+\varepsilon)^\lambda\int_{1-c}^{1+c} u^{\alpha-2}\,du.$$

Since, $\lim_{\lambda\to\infty}\lambda^k(m_\eta/(m_\eta+\varepsilon))^\lambda=0$, for any $k\in\mathbb{R}^+$ and

$$|T_\lambda(f\chi_{\delta,x};x)|\leqslant\left(\frac{m_\eta}{m_\eta+\varepsilon}\right)^\lambda\frac{M_1a(\lambda_1)}{m_\eta^{\lambda_1}}\int_{1-c}^{1+c}u^{\alpha-2}\,du,$$

the lemma follows.

Remark. The definition of D_Ω can be replaced by a slightly more general one,

$$D_\Omega=\{f: f \text{ is locally integrable, } \overline{\lim_{u\to\infty}}\,(f(u)/\Omega(u))<\infty \text{ and }$$
$$\overline{\lim_{u\to 0}}\,(f(u)/\Omega(u))<\infty\}.$$

Next, we state the following basic approximation theorem whose proof follows from Lemma 2.1 and Lemma 2.2.

THEOREM 2.1. *Let $G\in T(\mathbb{R}^+)$ and Ω be a bounding function for G If $f\in D_\Omega$ is continuous at a point $x\in\mathbb{R}^+$, there holds*

$$\lim_{\lambda\to\infty}T_\lambda(f;x)=f(x). \tag{2.5}$$

Further, if f is continuous on a open interval containing the closed interval $[a,b]$, (2.5) holds uniformly in $x\in[a,b]$.

3. CONVERGENCE IN THE COMPLEX DOMAIN

In this section we study the convergence of the operators T_λ in the complex domain. In the following theorem, we establish the convergence of $T_\lambda f$ for $z\in\Lambda_\Psi$ and $f\in D_\Omega^\Psi$ and regular in the interior of the set Λ_Ψ, Λ_Ψ^0.

THEOREM 3.1. *Let $G \in T_{\Psi}(\mathbb{R}^+)$, Ω be a Ψ-bounding function for G and $f \in D_{\Omega}^{\Psi}$ be regular in Λ_{Ψ}^{0}. Then*

$$\lim_{\lambda \to \infty} T_{\lambda}(f; z) = f(z), \tag{3.1}$$

uniformly on each compact subset of Λ_{Ψ}.

Proof. The proof of the theorem contains the following intermediate lemma:

LEMMA 3.1. *If $G \in T_{\Psi}(\mathbb{R}^+)$, Ω is a Ψ-bounding function for G and $f \in D_{\Omega}^{\Psi}$ is regular in Λ_{Ψ}, then for each $z \in \Lambda_{\Psi}$ there holds $T_{\lambda}(f; z) = T_{\lambda}(F_z; |z|)$ for all λ sufficiently large, where $F_z = f(u \exp i \arg z)$.*

Proof of the Lemma 3.1. If $z \in \mathbb{R}^+$, the result is trivial. Hence we assume that $z \in \Lambda\Psi \backslash \mathbb{R}^+$.

Let Γ denote the boundary of the subset $D = \{(r, \theta) \in A(\arg z): r_0 \leqslant r \leqslant R_0\}$ of Λ_{Ψ}, where r_0 and R_0 are positive numbers. It follows from Cauchy's theorem that

$$\frac{z^{\alpha-1}}{a(\lambda)} \int_{\Gamma} w^{-\alpha} f(w)\, G^{\lambda}(zw^{-1})\, dw = 0. \tag{3.2}$$

In view of the regularity of $f \in D_{\Omega}^{\Psi}$, there exists a positive constant M such that $|f(w)| < M\Omega(|w|)$ if $0 < \arg w \leqslant \arg z$. Hence if C denotes one of the two arcs of Γ,

$$\begin{aligned} &\left| \frac{z^{\alpha-1}}{a(\lambda)} \int_{C} w^{-\alpha} f(w)\, G^{\lambda}(zw^{-1})\, dw \right| \\ &\qquad \leqslant M \frac{|z|^{\alpha-1}}{a(\lambda)} \int_{C} |w|^{-\alpha}\, \Omega(|w|)|\, G^{\lambda}(zw^{-1})|\, |dw| \\ &\qquad \leqslant \frac{M_0 M(\lambda_0)}{a(\lambda)} \sup_{w \in C} |G(zw^{-1})|^{\lambda-\lambda_0}, \end{aligned} \tag{3.3}$$

for some positive constants M_0 and λ_0. Since $G \in T_{\Psi}(\mathbb{R}^+)$, $\sup_{w \in C} |G(zw^{-1})| \to 0$ as $r_0 \to 0$ and $R_0 \to \infty$. It follows that if $\lambda > \lambda_0$,

$$\begin{aligned} &\frac{a^{\alpha-1}}{a(\lambda)} \int_0^{\infty} u^{-\alpha} f(u)\, G^{\lambda}(zu^{-1})\, du \\ &\qquad = \frac{|z|^{\alpha-1}}{a(\lambda)} \int_0^{\infty} u^{-\alpha} f(ue^{i \arg z})\, G^{\lambda}(|z|\, u^{-1})\, du, \end{aligned} \tag{3.4}$$

which is just the relation $T_{\lambda}(f; z) = T_{\lambda}(F_z; |z|)$. This completes the proof of the Lemma 3.1.

Continuing the proof of the Theorem 3.1, we observe that $F_z(u) \in D_\Omega^\Psi$ and that it is continuous on $\mathbb{R}^+$. Since $F_z(|z|) = f(z)$, in view of Lemma 3.1 and Theorem 2.1, we have $\lim_{\lambda \to \infty} T_\lambda(f; z) = f(z)$, for each $z \in \Lambda_\Psi^0$. Moreover, the convergence is uniform on each closed segment of the ray $R_\Phi = \{z = re^{i\Phi}, r > 0\}$ $(0 \leqslant \Phi < \Psi)$.

Now let K be any compact subset of Λ_Ψ. It is clear that K can be enclosed in a region of type D considered in the proof of Lemma 3.1. In view of the uniform convergence of $T_\lambda(f; z)$ on closed segments of R_Φ $(0 < \Phi < \Psi)$, for all sufficiently large λ, $T_\lambda(f; z)$ is uniformly bounded on the linear segments of Γ. Thus the proof of Theorem 3.1 follows from Vitali's convergence theorem, [13, p. 168] provided we are able to establish uniform boundedness of $T_\lambda(f; z)$ for all λ sufficiently large and z belonging to the two arcs of Γ.

Let z be on one of these arcs. Then $f \in D_\Omega^\Psi$ and is regular in Λ_Ψ, for some $M > 0$ independent of z, we have

$$|T_\lambda(f; z)| = T_\lambda(F_z; |z|) \leqslant M T_\lambda(\Omega; |z|),$$

for all λ sufficiently large.

Applying Theorem 2.1 to the function Ω the required uniform boundedness follows for all λ sufficiently large. This completes the proof of Theorem 3.1.

In Theorem 3.1, f has been assumed to be regular in Λ_Ψ^0. The question arises as to what happens if f has certain singularities in Λ_Ψ^0. We consider this problem when f has a single isolated singularity in Λ_Ψ^0. In this case, it turns out that the convergence (3.1) may not hold throughout Λ_Ψ^0, and our interest lies in determining a subset of Λ_Ψ^0 on which the convergence takes place.

Let $G \in T_\Psi(\mathbb{R}^+)$. We define

$$\Lambda_\Psi^G = \{z \in \Lambda_\Psi : |G(z)| < G(1)\},$$

and denote the set $\{az : z \in \Lambda_\Psi^G\}$, a is a complex number, by $a\Lambda_\Psi^G$. It is clear that $\mathbb{R}^+ \setminus \{1\} \subset \Lambda_\Psi^G$.

THEOREM 3.2. *Let $G \in T_\Psi(\mathbb{R}^+)$ Ω be a Ψ-bounding function for G and $f \in D_\Omega^\Psi$. If f is regular in Λ_Ψ except for an isolated singularity w_0 in Λ_Ψ^0, there holds*

$$\lim_{\lambda \to \infty} T_\lambda(f; z) = f(z),$$

uniformly on compact subsets of $\Lambda_\Psi^0 - w_0 \Lambda_\Psi^G$.

Proof. Let K be a compact subset of $\Lambda_\Psi^0 - w_0\Lambda_\Psi^G$. Let C_r denote a circle with centre at w_0 and radius $r > 0$, We can choose r so small such that C_r does not interest $\Lambda_\Psi^0 - w_0\Lambda_\Psi^0$. If we define $F(w) = f(w)$ for w lying outside or on C_r and $F(w) = 0$ inside C_r and choose θ_0 such that $K \subset A(\theta_0)$, there exists an $M_1 > 0$ such that

$$|F(ue^{i\theta})| < M_1\Omega(u) \qquad \text{for all } 0 < \theta \leqslant \theta_0.$$

Let $z \in K$. Then, following the proof of Lemma 3.1, we have

$$\begin{aligned} &| T_\lambda(f; z) - T_\lambda(F(u \exp i \arg z); |z|) \\ &\quad \leqslant \frac{|z|^{\alpha-1}}{a(\lambda)} \int_{C_r} |w|^{-\alpha} |f(w)| |G(zw^{-1})|^\lambda |dw|. \end{aligned} \tag{3.5}$$

Since K is compact, there exists a positive number p such that $|G(zw^{-1})| < G(1) - 2p$, for each $z \in K$ and $w \in C_r$. Hence, there exists a constant M_2 such that

$$\begin{aligned} &| T_\lambda(f; z) - T_\lambda(F(u \in \exp i \arg z); |z|)| \\ &\quad \leqslant \frac{M_2}{a(\lambda)} (G(1) - 2p)^\lambda, \end{aligned} \tag{3.6}$$

for all $z \in K$. The right-hand side of this inequality approaches zero as $\lambda \to \infty$. For, we can find a $\delta \in (0, 1)$ such that $G(u) \geqslant G(1)$-p for all $u \in (1 - \delta, 1 + \delta)$ and then,

$$a(\lambda) \geqslant \delta(G(1) - p)^\lambda. \tag{3.7}$$

It follows that $T_\lambda(f; s) - T_\lambda(F(u \exp i \arg z); |z|)$ converges uniformly to zero for $z \in K$.

Let $a = \min\{|z| : z \in K\}$, $b = \max\{|z| : z \in K\}$. Now, $|T_\lambda(F(u \exp i \arg z); |z|)| \leqslant M_1 T_\lambda(\Omega; |z|)$, and in view of Theorem 2.1, the latter is uniformly bounded for $a \leqslant |z| \leqslant b$, and for all λ sufficiently large.

It follows that $T_\lambda(f; z)$ is uniformly bounded for $z \in K$ and all λ sufficiently large. But, again by Theorem 2.1, $T_\lambda(F(u \exp i \arg z); |z|) \to f(z)$ $(z \in K)$ as $\lambda \to \infty$ and hence also $T_\lambda(f; z) \to f(z)$ $(z \in K)$ as $\lambda \to \infty$. Now Vitali's convergence theorem is applicable and the proof of the theorem is complete.

Finally, we show that the region $\Lambda_\Psi^0 - w_0\Lambda_\Psi^G$ obtained in Theorem 3.2 is best possible, in a certain sence.

THEOREM 3.3. *Let $G \in T_\Psi(\mathbb{R}^+)$, Ω be a Ψ-bounding function for G, and $w_0 \in \Lambda_\Psi^0$. Then there exists a function $f \in D_\Omega^\Psi$ whose only singularity in Λ_Ψ^0 is w_0 and for which $T_\lambda(f; z)$ diverges for each $z \in w_0\Lambda_\Psi^G - \{w_0\}$.*

Proof. The function $f(z)=(z-w_0)^{-1}\in D_\Omega^\Psi$ for each Ψ-bounding function for G. Also, if $z\in w_0 \Lambda_\Psi^G - \{w_0\}$, $\arg z > \arg w_0$. Hence by Cauchy's theorem, as in the proof of Lemma 3.1, we have

$$T_\lambda(f; z) = T_\lambda(u \exp i \arg z - w_0)^{-1}; |z|) + \frac{z^{\alpha-1}}{a(\lambda)} \int_{C_r} \frac{w^{-\alpha}}{w-w_0} G^\lambda(zw^{-1})\, dw, \tag{3.8}$$

where C_r is as in the proof of Theorem 3.2, with r sufficiently small. By the residue theorem,

$$T_\lambda(f; z) = T_\lambda((u \exp i \arg z - w_0)^{-1}; |z|) + \frac{2\pi i}{a(\lambda)} w_0^{-\alpha} G^\lambda(zw_0^{-1}). \tag{3.9}$$

In view of Theorem 2.1, the first term on the right-hand side of (3.9) converges to $(z-w_0)^{-1}$. But, since $z\in w_0\Lambda_\Psi^G$, $|G(zw_0^{-1})| > G(1)$. In view of Lemma 2.1, it follows that $T_\lambda(f; z)$ diverges as $\lambda \to \infty$. This completes the proof of the theorem.

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